Math 1020 Week 1

Vectors

A vector is something with

① magnitude (length)

② direction

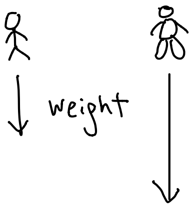
Notation vector : $\vec{V}$, $\mathbf{V}$, $\bar{V}$

Length : $|\vec{v}| = \|\vec{v}\|$

Physics

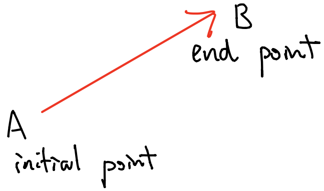
e.g. velocity

e.g. Force



Geometrically

Vector can be represented by an arrow

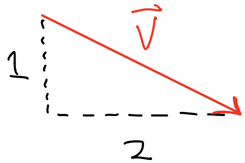


$\overrightarrow{AB}$ = vector from A to B

length = magnitude

Algebraically

e.g. $\vec{V} = (2,-1)$ in $\mathbb{R}^2 = \mathbb{R}\times\mathbb{R}$ (2D)



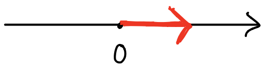
going to the right by 2 units

going up by -1 unit

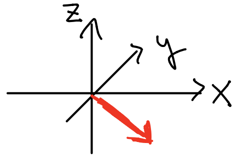
i.e down by 1 unit

We can talk about in other dimensions too

e.g.



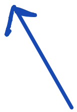
(2) in 1D



$(1,3,-4)$ in 3D

Basic operations

e.g. $\vec{v} = (1,1)$ $\qquad$ $\vec{w} = (-1,2)$



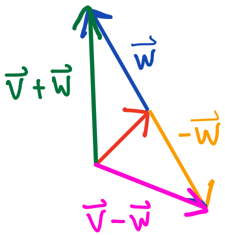
Addition $\quad \vec{v} + \vec{w} = (1+(-1), 1+2) = (0,3)$

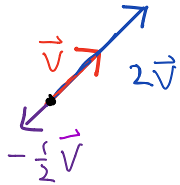
Subtraction $\quad \vec{v} - \vec{w} = (1-(-1), 1-2) = (2,-1)$

Scalar multiplication $\quad 2\vec{v} = (2(1), 2(1)) = (2,2)$

$$-\tfrac{1}{2}\vec{v} = \left(-\tfrac{1}{2}(1), -\tfrac{1}{2}(1)\right) = \left(-\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

Scalar means number





Defn Define $\vec{0}$ = zero vector

$$= \begin{cases} (0,0) & \text{in 2D} \\ (0,0,0) & \text{in 3D} \end{cases}$$

Rmk Similar in other dimensions

Properties Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, $\alpha, \beta \in \mathbb{R}$

1. $0\,\vec{v} = \vec{0}$

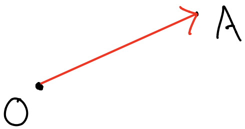
   number zero $\qquad$ zero vector

2. $1\vec{v} = \vec{v}$
3. $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$ $\quad$ Associative
4. $\vec{v}+\vec{w} = \vec{w}+\vec{v}$ $\quad$ Commutative
5. $\vec{v}+\vec{0} = \vec{v}$
6. $(\alpha+\beta)\vec{v} = \alpha\vec{v} + \beta\vec{v}$
7. $\alpha(\vec{v}+\vec{w}) = \alpha\vec{v} + \alpha\vec{w}$

   (6)–(7): Distributive

8. $(\alpha\beta)\vec{v} = \alpha(\beta\vec{v})$

A **Position Vector** is a vector with initial point the origin

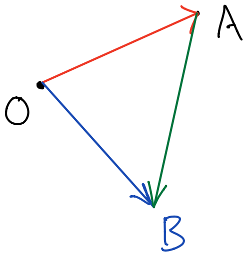


Given points $A=(x_1,y_1)$ $B=(x_2,y_2)$

Position vectors :

$\overrightarrow{OA} = (x_1, x_2)$ $\quad \overrightarrow{OB} = (x_2, y_2)$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{AO} + \overrightarrow{OB} \\ &= -\overrightarrow{OA} + \overrightarrow{OB} \\ &= -(x_1, y_1) + (x_2, y_2) \\ &= (x_2 - x_1, y_2 - y_1)\end{aligned}$$



eg. $A=(1,0)$, $B=(3,3)$ $C=(2,4)$ $D=(0,1)$

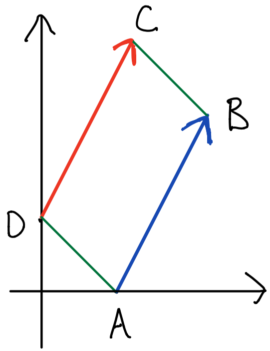
Show that ABCD is a parallelogram

Sol

$\overrightarrow{AB}$ = vector from A to B

$$\begin{aligned}&= (3,3)-(1,0) \\ &= (2,3)\end{aligned}$$

$$\begin{aligned}\overrightarrow{DC} &= (2,4)-(0,1) \\ &= (2,3)\end{aligned}$$



$\Rightarrow \overrightarrow{AB} = \overrightarrow{DC}$

$\Rightarrow$ ABCD is a parallelogram

Rmk $\overrightarrow{AB}$ and $\overrightarrow{DC}$ are considered equal as they have same magnitude and direction even though with different initial points

Standard vectors

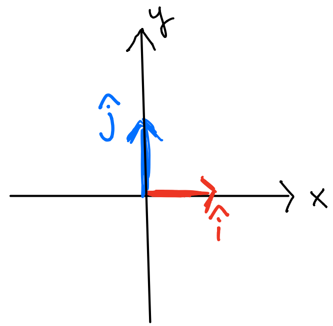
2D $\hat{i} = (1,0)$

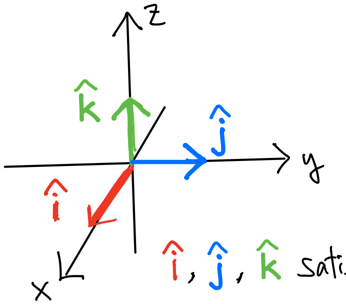
$\hat{j} = (0,1)$

3D $\hat{i} = (1,0,0)$

$\hat{j} = (0,1,0)$

$\hat{k} = (0,0,1)$





$\hat{i}, \hat{j}, \hat{k}$ satisfy right hand rule

Rmk

① We use $\wedge$ (hat) instead of $\rightarrow$ (arrow) to emphasize that $\hat{i}, \hat{j}, \hat{k}$ have length 1

② Standard vectors form a "building unit" of other vectors, eg:

$$(1,-2,3) = (1,0,0) - (0,2,0) + (0,0,3)$$
$$= \hat{i} - 2\hat{j} + 3\hat{k}$$

Length and Dot Product

2D $\vec{a} = (a_1, a_2)$ $\quad \vec{b} = (b_1, b_2)$

Length $\quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2}$

Dot Product $\quad \vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$

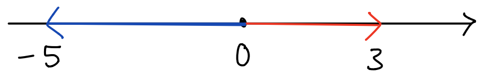
Do not forget the dot

Rmk Similar in other dimension

eg. 3D $\quad \|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

eg. 1D $\quad \|a\| = |a| =$ absolute value !



$|3| = \sqrt{3^2} = 3$ ← Distance from

$|-5| = \sqrt{(-5)^2} = 5$ ← origin

Properties of $\|\vec{v}\|$

① $\|\vec{v}\| \geq 0$

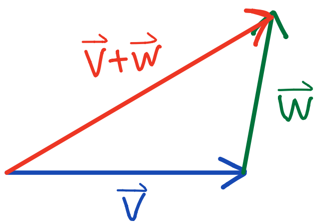
② $\|\vec{v}\| = 0 \iff \vec{v} = 0$

③ $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

④ $\|\alpha \vec{v}\| = |\alpha| \, \|\vec{v}\|$, α is a scalar

⑤ $\|\vec{v} + \vec{w}\| \leq \|\vec{v}\| + \|\vec{w}\|$

(Triangle inequality)



Def A unit vector $\vec{v}$ is a vector with length 1. We sometimes use "hat" $\hat{v}$ for unit vector.

eg let $\vec{v} = (1, 2, -2)$

i. Find $\|\vec{v}\|$

ii. Find the unit vector in the opposite direction of $\vec{v}$

Sol

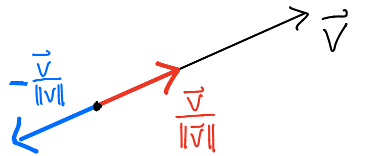
i. $\|\vec{v}\| = \sqrt{1^2 + 2^2 + (-2)^2} = 3$

ii Unit vector in the same direction of $\vec{v}$:

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{3}\vec{v} = \left(\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right)$$

Unit vector in the opposite direction of $\vec{v}$:

$$-\frac{\vec{v}}{\|\vec{v}\|} = -\frac{1}{3}\vec{v} = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$



Properties of Dot Product

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, $\alpha, \beta \in \mathbb{R}$

① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$

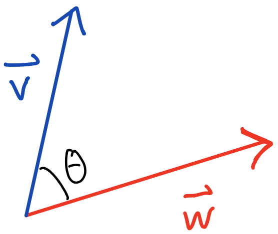
② $(\alpha \vec{u} + \beta \vec{v}) \cdot \vec{w} = \alpha\, \vec{u} \cdot \vec{w} + \beta\, \vec{v} \cdot \vec{w}$

③ $\vec{0} \cdot \vec{v} = 0$

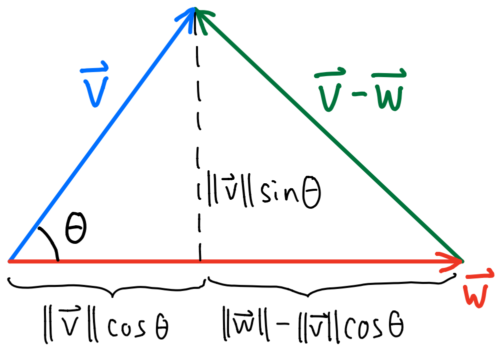
④ $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$

⑤ $\vec{v} \cdot \vec{w} = \|\vec{v}\|\, \|\vec{w}\| \cos\theta$, where

θ is the angle between v and w



Pf of ⑤ (Also proof of cosine law)



Note

$$\|\vec{v} - \vec{w}\|^2 = (\|\vec{v}\| \sin\theta)^2 + (\|\vec{w}\| - \|\vec{v}\| \cos\theta)^2 \quad \text{(Pythagoras thm)}$$

$$= \|\vec{v}\|^2 \sin^2\theta + \|\vec{w}\|^2 - 2\|\vec{w}\|\,\|\vec{v}\| \cos\theta + \|\vec{v}\| \cos^2\theta$$

$$= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\,\|\vec{v}\|\,\|\vec{w}\| \cos\theta \quad \text{(I)}$$

Also,

$$\|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w}) \cdot (\vec{v} - \vec{w})$$

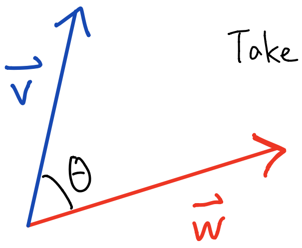
$$= \vec{v} \cdot \vec{v} - \vec{w} \cdot \vec{v} - \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{w}$$

$$= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\,\vec{v} \cdot \vec{w} \quad \text{(II)}$$

Compare (I) and (II) $\Rightarrow$ ⑤

Formula

$$\boxed{\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta}$$



Take $0^\circ \le \theta \le 180^\circ$

If $\vec{v}, \vec{w} \ne \vec{0}$,

then $\|\vec{v}\|, \|\vec{w}\| > 0$, $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

$$\Rightarrow \begin{cases} \vec{v}\cdot\vec{w} > 0 \iff 0^\circ \le \theta < 90^\circ & \text{acute angle} \\ \vec{v}\cdot\vec{w} = 0 \iff \theta = 90^\circ & \text{right angle} \\ \vec{v}\cdot\vec{w} < 0 \iff 90^\circ < \theta \le 180^\circ & \text{obtuse angle} \end{cases}$$

$$\boxed{\vec{v} \perp \vec{w} \iff \vec{v}\cdot\vec{w} = 0}$$

eg Find angle θ between

$$\vec{v} = \hat{i} + \hat{j} + 3\hat{k}, \quad \vec{w} = 2\hat{i} - \hat{j} - 2\hat{k}$$

$$\boxed{\begin{array}{lll} \hat{i}\cdot\hat{i} = 1 & & \\ \hat{i}\cdot\hat{j} = 0 & \hat{j}\cdot\hat{j} = 1 & \\ \hat{i}\cdot\hat{k} = 0 & \hat{j}\cdot\hat{k} = 0 & \hat{k}\cdot\hat{k} = 1 \end{array}}$$

Sol $\vec{v}\cdot\vec{w} = (\hat{i} + \hat{j} + 3\hat{k})\cdot(2\hat{i} - \hat{j} - 2\hat{k})$

$= (1)(2) + (1)(-1) + (3)(-2) = -5$

$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$

$\|\vec{w}\| = \sqrt{\vec{w}\cdot\vec{w}} = \sqrt{2^2 + (-1)^2 + (-2)^2} = 3$

$\therefore \cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \dfrac{-5}{\sqrt{11}\,(3)}$

$\Rightarrow \theta = \arccos\left(\dfrac{-5}{3\sqrt{11}}\right) \approx 120.17^\circ$

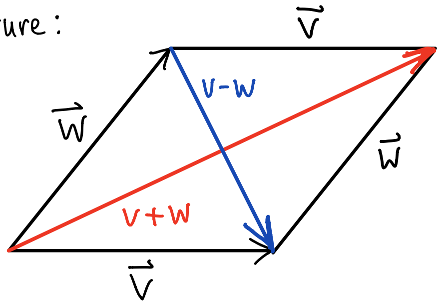
eg Let $\vec{v}, \vec{w}$ have same length

Show that $(\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) = 0$

Sol

$$\begin{aligned} \text{L.H.S.} &= (\vec{v}+\vec{w})\cdot(\vec{v}-\vec{w}) \\ &= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{v} - \vec{w}\cdot\vec{w} \\ &= \|v\|^2 - \vec{v}\cdot\vec{w} + \vec{v}\cdot\vec{w} - \|w\|^2 \\ &= \|v\|^2 - \|v\|^2 \quad (\text{same length}) \\ &= 0 \end{aligned}$$

Picture:



$\|v\| = \|w\|$

$\Rightarrow$ rhombus

(菱形)

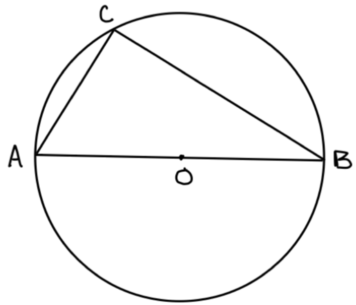
Geometric meaning

Diagonals of a rhombus are perpendicular

eg Consider a circle centered at O.

AB is diameter.

Show that $\angle ACB = 90°$



Sol

$$\vec{AC} = \vec{AO} + \vec{OC}$$

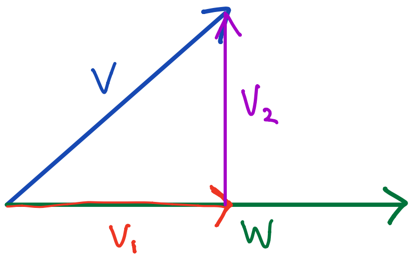
$$\vec{BC} = \vec{BO} + \vec{OC} = -\vec{AO} + \vec{OC}$$

$$\begin{aligned} \vec{AC}\cdot\vec{BC} &= (\vec{AO}+\vec{OC})\cdot(-\vec{AO}+\vec{OC}) \\ &= -\vec{AO}\cdot\vec{AO} + \vec{AO}\cdot\vec{OC} - \vec{OC}\cdot\vec{AO} + \vec{OC}\cdot\vec{OC} \\ &= -\|\vec{AO}\|^2 + \|\vec{OC}\|^2 \quad \left(\begin{array}{c} \|\vec{AO}\| = \|\vec{OC}\| \\ \text{are radius} \end{array}\right) \\ &= 0 \end{aligned}$$

$\therefore \vec{AC} \perp \vec{BC} \Rightarrow \angle ACB = 90°$

Projection Vector

Given vectors $\vec{v}, \vec{w}$ with $\vec{w} \neq \vec{0}$.



Decompose $\vec{v}$ as $\vec{v} = \vec{v}_1 + \vec{v}_2$ such that

① $\vec{v}_1 \parallel \vec{w}$ (ie. v_1, w have same/opposite directions)

② $\vec{v}_2 \perp \vec{w}$

Define

$$\vec{v}_1 = \mathrm{Proj}_{\vec{w}} \vec{v}$$

$$= \text{Projection of } \vec{v} \text{ onto } \vec{w}$$

We want to find a formula for $\mathrm{Proj}_{\vec{w}} \vec{v}$

Observations

① $\vec{v}_1 \parallel \vec{w} \Rightarrow \vec{v}_1 = k\vec{w}$ for some $k \in \mathbb{R}$

② $\vec{v}_2 \perp \vec{w} \Rightarrow \vec{v}_2 \cdot \vec{w} = 0$

$$\Rightarrow (\vec{v} - \vec{v}_1) \cdot \vec{w} = 0$$

$$\Rightarrow \vec{v} \cdot \vec{w} = \vec{v}_1 \cdot \vec{w} = (k\vec{w}) \cdot \vec{w} = k(\vec{w} \cdot \vec{w})$$

$$\Rightarrow k = \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}$$

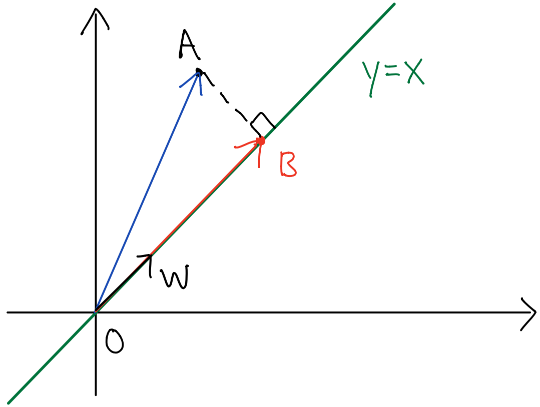
$$\therefore \quad \boxed{\mathrm{Proj}_{\vec{w}} \vec{v} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \right) \vec{w} = \frac{\vec{v} \cdot \vec{w}}{\|\vec{w}\|^2} \vec{w}}$$

Warning

$$\cancel{\frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} = \frac{\vec{v}}{\vec{w}}}$$

Cannot cancel $\vec{w}$ in quotient of dot product eg Find the point B on the line $L: y=x$ which is closest to $A=(3,7)$.

Sol Let $\vec{w}=(1,1)$, $\vec{w} \parallel L$



$$\overrightarrow{OB} = \text{Proj}_{\vec{w}} \overrightarrow{OA}$$

$$= \left(\frac{\overrightarrow{OA}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}$$

$$= \frac{(3,7)\cdot(1,1)}{(1,1)\cdot(1,1)}(1,1)$$

$$= \frac{10}{2}(1,1)$$

$$= (5,5)$$

$\therefore B=(5,5)$

Determinant

Define $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

2×2 determinant

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

3×3 determinant

eg. $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (2)(3) = -2$

eg $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = (1) \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - (2) \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (3) \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$

$= (1)\left[(5)(9) - (6)(8)\right] - (2)\left[(4)(9) - (6)(7)\right] + 3\left[(4)(8) - (5)(7)\right]$

$= -3 + 12 - 9$

$= 0$

Cross Product (only in 3D)

Let $\vec{a} = (a_1, a_2, a_3)$ $\vec{b} = (b_1, b_2, b_3)$

Define the cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k$$

$$= (a_2 b_3 - a_3 b_2, a_3 b_1 - a_1 b_3, a_1 b_2 - a_2 b_1)$$

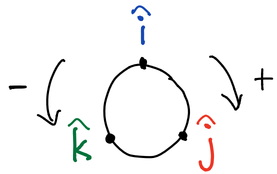
eg

$$\hat{i} \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} \hat{i} - \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} \hat{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \hat{k}$$

$$= (0)\hat{i} - (0)\hat{j} + (1)\hat{k} = \hat{k}$$

$$\boxed{\begin{array}{lll} \hat{i} \times \hat{i} = \vec{0} & \hat{i} \times \hat{j} = \hat{k} & \hat{i} \times \hat{k} = -\hat{j} \\ \hat{j} \times \hat{i} = -\hat{k} & \hat{j} \times \hat{j} = \vec{0} & \hat{j} \times \hat{k} = \hat{i} \\ \hat{k} \times \hat{i} = \hat{j} & \hat{k} \times \hat{j} = -\hat{i} & \hat{k} \times \hat{k} = \vec{0} \end{array}}$$



eg Let $\vec{a} = 2\hat{i} + 3\hat{j} + 5\hat{k}$, $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 5 \\ 1 & 2 & 3 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 5 \\ 2 & 3 \end{vmatrix} \hat{i} - \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} \hat{j} + \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \hat{k}$$

$$= -\hat{i} - \hat{j} + \hat{k}$$

Ex Find $\vec{b} \times \vec{a}$ and $\vec{b} \times \vec{b}$.

Properties of Cross Product

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors in $\mathbb{R}^3$, $\alpha, \beta \in \mathbb{R}$

Algebraic

1. $\vec{a} \times \vec{a} = \vec{0}$

2. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

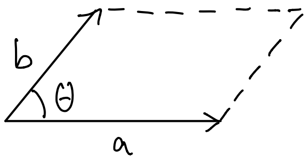
3. $(\alpha\vec{a} + \beta\vec{b}) \times \vec{c} = \alpha\vec{a} \times \vec{c} + \beta\vec{b} \times \vec{c}$

Geometric

4. Let θ be the angle between $\vec{a}, \vec{b}$

$$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$$

$= $ Area of ▱ spanned by $\vec{a}, \vec{b}$



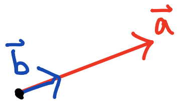
$0 \leq \theta \leq \pi$

$\therefore \quad \vec{a} \times \vec{b} = \vec{0}$

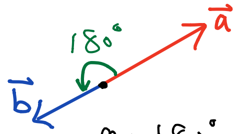
$\iff \|\vec{a} \times \vec{b}\| = 0$

$\iff \|a\|$ or $\|b\|$ or $\sin\theta = 0$

$\iff \vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\theta = 0^\circ$ or $\theta = 180^\circ$



$\theta = 0^\circ$



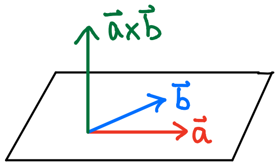
$\theta = 180^\circ$

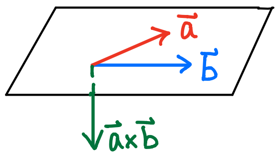
5. If $\vec{a} \times \vec{b}$ is non-zero, then

$$(\vec{a} \times \vec{b}) \cdot \vec{a} = (\vec{a} \times \vec{b}) \cdot \vec{b} = 0$$

$$\Rightarrow \vec{a} \times \vec{b} \perp \vec{a}, \quad \vec{a} \times \vec{b} \perp \vec{b}$$

Also, $\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ satisfy right hand rule

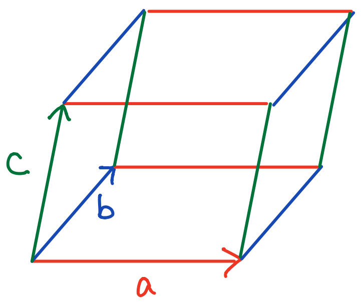




6. The triple product of $\vec{a}, \vec{b}, \vec{c}$ is defined to be

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$|\vec{a}\cdot(\vec{b}\times\vec{c})|$ = Volume of parallelepiped spanned by $\vec{a}, \vec{b}, \vec{c}$



Rmk For $\vec{a}, \vec{b}, \vec{c} \in \mathbb{R}^3$,

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= -\vec{a}\cdot(\vec{c}\times\vec{b}) = -\vec{b}\cdot(\vec{a}\times\vec{c}) = -\vec{c}\cdot(\vec{b}\times\vec{a})$$

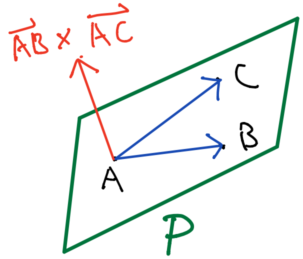
(Can be proved easily using property of determinant discussed later)

eg Let $A=(1,2,1)$, $B=(1,-1,0)$, $C=(2,3,2)$ be points on a plane P.

Find a vector which is perpendicular to P

Sol $\overrightarrow{AB} = (1,-1,0)-(1,2,1)$
$= (0,-3,-1)$

$\overrightarrow{AC} = (2,3,2)-(1,2,1)$
$= (1,1,1)$



$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -3 & -1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} -3 & -1 \\ 1 & 1 \end{vmatrix}\hat{i} - \begin{vmatrix} 0 & -1 \\ 1 & 1 \end{vmatrix}\hat{j} + \begin{vmatrix} 0 & -3 \\ 1 & 1 \end{vmatrix}\hat{k}$$

$$= [(-3)(1)-(-1)(1)]\hat{i} - [(0)(1)-(-1)(1)]\hat{j} + [(0)(1)-(-3)(1)]\hat{k}$$

$$= -2\hat{i} - \hat{j} + 3\hat{k}$$

$\therefore (-2,-1,3) \perp P$.

eg. Let A, B, C, D, E be points in 3D with $A = (2,2,1)$, $B = (1,1,1)$
$C = (1,2,0)$, $E = (2,4,-5)$

Suppose ABCD is a parallelogram.

a. Find coordinates of D

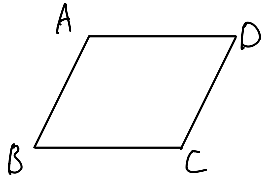
b. Find area of ▱ABCD

c. Find a unit vector $\perp$ ▱ABCD

d. Find the equation of the plane containing A, B, C

e. Find volume of parallelepiped with adjacent sides $\overrightarrow{BA}$, $\overrightarrow{BC}$ and $\overrightarrow{BE}$

Sol



a. ABCD is ▱

$$\overrightarrow{CD} = \overrightarrow{BA} = (2,2,1) - (1,1,1) = (1,1,0)$$

$$C = (1,2,0)$$

$$\Rightarrow D = (1,2,0) + (1,1,0) = (2,3,0)$$

b. $\overrightarrow{BC} = (1,2,0) - (1,1,1) = (0,1,-1)$

$$\overrightarrow{BA} \times \overrightarrow{BC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 0 \\ 0 & 1 & -1 \end{vmatrix} = -\hat{i} + \hat{j} + \hat{k}$$

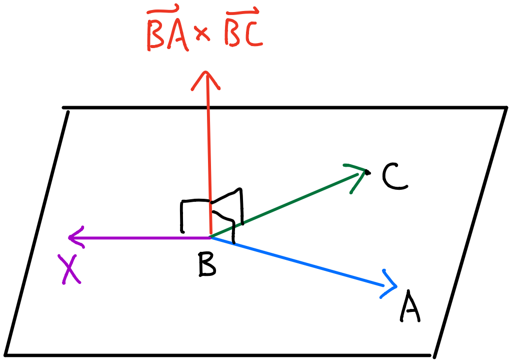
Area of ▱ABCD $= \|\overrightarrow{BA} \times \overrightarrow{BC}\|$

$$= \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$$

c. $\overrightarrow{BA} \times \overrightarrow{BC} \perp$ ▱

Required vector $= \dfrac{\overrightarrow{BA} \times \overrightarrow{BC}}{\|\overrightarrow{BA} \times \overrightarrow{BC}\|} = \dfrac{1}{\sqrt{3}}(-i + j + k)$

d. Suppose $X=(x,y,z)$ is on the plane



$\overrightarrow{BA}\times\overrightarrow{BC} \perp$ the plane

$\Rightarrow \overrightarrow{BA}\times\overrightarrow{BC} \perp \overrightarrow{BX}$

$\Rightarrow (\overrightarrow{BA}\times\overrightarrow{BC})\cdot\overrightarrow{BX} = 0$

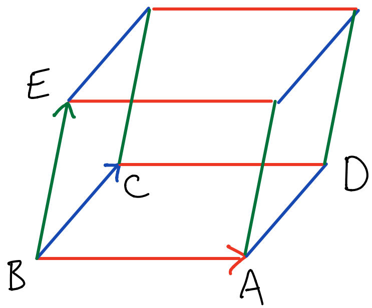
$(-1,1,1)\cdot[(x,y,z)-(1,1,1)]=0$

$(-1,1,1)\cdot(x-1,y-1,z-1)=0$

$-(x-1)+(y-1)+(z-1)=0$

$\therefore\ -x+y+z-1=0$

e.



$\overrightarrow{BE}=(2,4,-5)-(1,1,1)=(1,3,-6)$

Triple product $= \overrightarrow{BE}\cdot(\overrightarrow{BA}\times\overrightarrow{BC})$

$= (1,3,-6)\cdot(-1,1,1)$

$= -4$

$\therefore$ Volume of parallelepiped $= |-4| = 4$

Vector-valued function (Parametric Equation)

Most functions you saw are real-valued:

e.g.

$f(x) = x^2 + 2$ ← real-valued function

$f(0) = 2$ ← outputs are real numbers

$f(5) = 27$

However, one can also consider functions which are vector-valued:

eg

$\vec{r}(t) = (t^2, 2t-1)$ ← vector-valued function (2D)

$= t^2 \hat{i} + (2t-1)\hat{j}$ (Another form)

then $\vec{r}(2) = (4, 3)$ ← Outputs are vectors

$= 4\hat{i} + 3\hat{j}$

Rmk ① To understand $\vec{r}(t)$, sometimes it is useful to regard t = time and $\vec{r}(t)$ = displacement of an object at time t

② In the example above, we can define

$x(t) = t^2$, $y(t) = 2t-1$

then $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$x(t)$, $y(t)$ are called <u>component functions</u> of $\vec{r}(t)$.

t is called <u>parameter</u>

Vector-valued functions in 3D

eg $\vec{r}(t) = (e^t, t, \sqrt{1+t^2})$

$= e^t \hat{i} + t\hat{j} + \sqrt{1+t^2}\,\hat{k}$

It has component functions

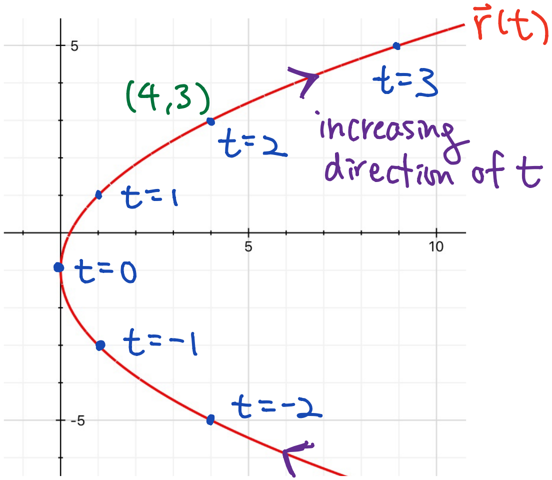
$x(t) = e^t$ $\quad y(t) = t$ $\quad z(t) = \sqrt{1+t^2}$

Graphing a vector-valued function

We can graph $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ on the xy-plane:

eg $\vec{r}(t) = t^2\hat{i} + (2t-1)\hat{j}$, then

t	-2	-1	0	1	2
$x(t)$	4	1	0	1	4
$y(t)$	-5	-3	-1	1	3



<u>Q</u> How to plot $\vec{r}(t)$ above accurately?

<u>A</u> Note $x = x(t) = t^2$

$$y = y(t) = 2t-1 \Rightarrow t = \frac{y+1}{2}$$

$$\therefore x = \left(\frac{y+1}{2}\right)^2 = \frac{1}{4}(y+1)^2 \quad \text{(Parabola)}$$

<u>Q</u> Graph $\vec{r}(t) = (2\cos t^\circ)\hat{i} + (2\sin t^\circ)\hat{j}$ for $0 \le t \le 180$.

<u>Sol</u> $x = 2\cos t^\circ \quad y = 2\sin t^\circ$

$$\therefore x^2 + y^2 = (2\cos t^\circ)^2 + (2\sin t^\circ)^2$$
$$= 4(\cos^2 t^\circ + \sin^2 t^\circ)$$
$$= 4$$

$\therefore \vec{r}(t)$ lies on the circle $x^2 + y^2 = 4$

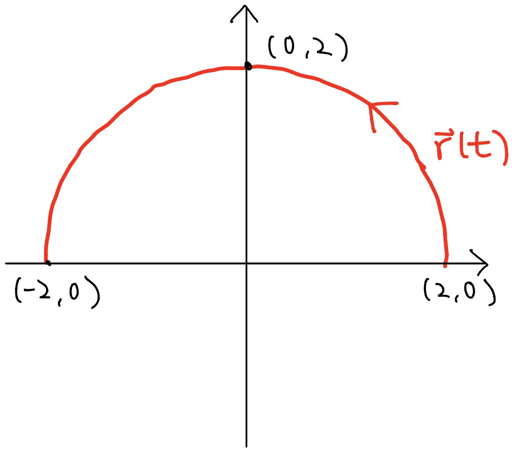
Also, as t increases from 0 to 180,

$x(t)$ decreases from 2 to -2

$y(t)$ increases from 0 to 2 and then decreases from 2 to 0

$\therefore$ Graph of

$$\vec{r}(t) = (2\cos t^\circ)\hat{i} + (2\sin t^\circ)\hat{j}$$

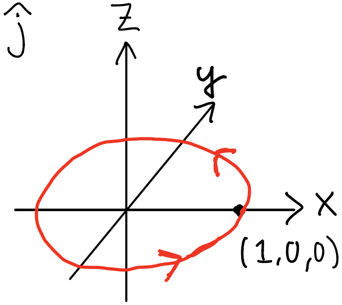


eg Let $\vec{r}(t) = (\cos 360 t^\circ)\hat{i} + (\sin 360 t^\circ)\hat{j} + t\hat{k}$

Plot the 3D graph of $\vec{r}(t)$ for $0 \leq t \leq 1$

Sol Note that as t increases from 0 to 1,

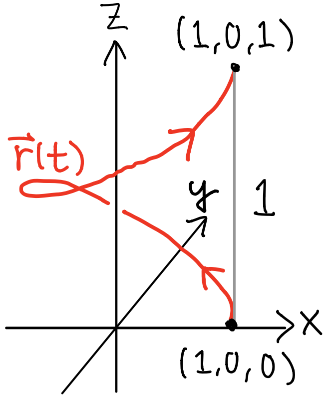
- $(\cos 360 t^\circ)\hat{i} + (\sin 360 t^\circ)\hat{j}$ rotates around the origin along the unit circle on the xy-plane



- $z(t) = t$ increases from 0 to 1

$\Rightarrow$ $\vec{r}(t)$ moves up from xy-plane to $z=1$

$\therefore$ $\vec{r}(t)$ is a "helix"



eg. Plot $\vec{r}(t) = (2+t)\hat{i} + (4-2t)\hat{j} - \hat{k}$.

<u>Sol</u> Note that

$$\vec{r}(t) = (2\hat{i} + 4\hat{j} - \hat{k}) + t(\hat{i} - 2\hat{j})$$

$\vec{r}(t)$ is the straight line parallel to $\hat{i} - 2\hat{j}$ and passes through $(2, 4, -1)$